ON NONLINEAR FUNCTIONALS OF RANDOM SPHERICAL EIGENFUNCTIONS

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Abstract. We prove Central Limit Theorems and Stein-like bounds for the asymptotic behaviour of nonlinear functionals of spherical Gaussian eigenfunctions. Our investigation combine asymptotic analysis of higher order moments for Legendre polynomials and, in addition, recent results on Malliavin calculus and Total Variation bounds for Gaussian subordinated fields. We discuss application to geometric functionals like the Defect and invariant statistics, e.g. polyspectra of isotropic spherical random fields. Both of these have relevance for applications, especially in an astrophysical environment.

- **Keywords and Phrases:** Gaussian Eigenfunctions,High Energy Asymptotics, Central Limit Theorem, Total Variation Bounds
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1. Introduction

1.1. **Motivation.** Much effort has been recently devoted to the analysis of spherical Gaussian eigenfunctions (to be defined below). These random fields are the Fourier components in spectral representation expansions for general spherical Gaussian fields, see for instance [1], [9], [13], and [8], [11] for extensions; in view of this, their study is of obvious relevance in connection with the statistical analysis of spherical data. Namely, the analysis of these components and their polynomial transforms (the so-called *polyspectra*) is now one of the leading themes in modern Cosmology, in particular in the growing area of Cosmic Microwave Background (CMB) data analysis: we refer for instance to [2], [7], [10], [13] and the references therein.

In short and somewhat vague terms, CMB represents a relic electromagnetic radiation from the so-called *age of recombination*, e.g. the cosmological epoch when free electrons and protons combined to form stable hydrogen atoms; this is now reckoned to have occurred some 3.7×10^5 years after the Big Bang, i.e., some 1.3×10^{10} years ago. As such, CMB radiation is universally recognized as a goldmine of information on primordial epochs, and its analysis has drawn enormous theoretical and experimental efforts: we refer for instance to http://map.gsfc.nasa.gov/ and http://www.rssd.esa.int/Planck for the most relevant ongoing experimental activity.

Among many, one of the leading current themes is the analysis of non-Gaussianity of CMB data; for this purpose much effort has been dedicated

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to the investigation of the properties of the polyspectra mentioned above, e.g. polynomial transforms of sample spherical Fourier components. In this paper we provide a full characterization for the asymptotic behaviour of these transforms, proving Central Limit Theorem results under rather broad assumptions.

From a different perspective, the analysis of high frequency (or high energy) Laplace eigenfunctions is a classical subject in Analysis and Mathematical Physics with much progress in understanding, both rigorously and conjecturally, in the recent years. According to Berry's universality conjecture [3], the *deterministic* high energy eigenfunctions on "generic" manifolds are represented by *random* monochromatic waves, an equivalent planar model to our spherical Gaussian eigenfunctions.

One in particular interesting aspect of those is their *nodal structure* [5] (e.g. the zero set and the number of its connected components or ovals), especially in light of their conjectural relations to percolation and SLE [6]. Some results on the geometry of the nodal structure include studying the number of the ovals (equivalently the number of *nodal components* i.e. connected components of the zero set complement) [16], and the total length of the nodal line [26]. The so-called Defect (i.e. the difference between "hot" and "cold" regions for the eigenfunctions, see below or Section 4.2) of spherical Gaussian eigenfunctions was addressed in our earlier work [15].

1.2. **Statement of the main results.** Let $\Delta_{\mathbb{S}^2}$ denote the usual Laplacian on the 2-dimensional unit sphere $\mathcal{S}^2$, and consider the Gaussian random spherical eigenfunctions f_l, $l \in \mathbb{Z}_{>0}$, satisfying

$$\Delta_{\mathbb{S}^2} f_l = -l(l+1)f_l \ , \ \mathbb{E}\left[f_l(x)\right] = 0 \ , \ \mathbb{E}\left[f_l(x)^2\right] = 1.$$

It is well known that f_l can be expanded into the orthonormal basis of spherical harmonics $\{Y_{lm}\}$

$$f_l = \sum_{m=-l}^{l} a_{lm}Y_{lm} \ , \ \mathbb{E}\left[a_{lm}\overline{a}_{lm'}\right] = \frac{4\pi}{2l+1}\delta_m^{m'} ,$$

(where the coefficients a_{lm} Gaussian i.i.d.). The random field f_l is isotropic or rotation invariant, meaning that for any rotation $g \in SO(3)$ the distribution of $f_l(\cdot)$ is equal to the distribution of $f_l(g\,\cdot)$ (equivalently, for every $k \in \mathbb{N}$ and $x_1, \dots, x_k \in \mathbb{S}^2$, the distribution of the random vector $(f_l(x_1), \dots, f_l(x_k))$ equals to the distribution of $(f_l(g\cdot x_1), \dots, f_l(g\cdot x_k))$). Equivalently, f_l is centred Gaussian, with the covariance function

$$r_l(x,y) := \mathbb{E}[f_l(x)\cdot f_l(y)] = P_l(\langle x,y\rangle) \ , \ x,y \in S^2,$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product, so that $\vartheta(x,y) := \arccos(\langle x,y\rangle)$ is the angular distance; here, $P_l : [-1,1] \to \mathbb{R}$ are the usual Legendre polynomials, see the Appendix for more details.

The purpose of this paper is to study the asymptotic behaviour of random quantities such as

$$h_{l;q} = \int_{S^2} H_q(f_l(x))dx \ ,$$

where $H_q(.)$ are the Hermite polynomials satisfying the differential equations

$$H_m'(x) = mH_{m-1}(x) \ , \ \mathbb{E}\left[H_m(Z)\right] = 0 \ , \ Z \sim \mathcal{N}(0,1) \ .$$

As mentioned above, these nonlinear transforms are of interest by themselves as statistical functionals, in connections to the analysis of angular polyspectra of spherical random fields (more details to be provided below). Importantly, these statistics are the basic building blocks for the analysis of general nonlinear functionals of Gaussian eigenfunctions. More precisely, it is a well-known general fact (see for instance [19]) that the L^2 space of square-integrable nonlinear transforms of Gaussian eigenfunctions can be expanded (in the L^2 sense) as

(1)
$$G(f_l) = \sum_{q=0}^{\infty} \frac{J_q(G)}{q!} H_q(f_l) \ , \ \mathbb{E}\left[G^2(f_l)\right] < \infty \ , \ J_q(G) := \mathbb{E}\left[G(f_l)H_q(f_l)\right] \ .$$

As a consequence, the analysis of averaged statistics of the form

$$\mathcal{G}(f_l) = \int_{S^2} G(f_l)dx$$

for "generic" G will directly follow from Central Limit Theorem results on $\{h_{l;q}\}$. To establish the latter, we first need to investigate the asymptotic behaviour, as $l \to \infty$, for the variances $Var(h_{l;q})$. Note that when both q and l are odd, the statistics $\{h_{l;q}\}$ are identically zero for the symmetry properties of Legendre polynomials, e.g.

$$P_l(x) = (-1)^l P_l(-x),$$

whence integrals of odd polynomials over the sphere are identically zero. To simplify the discussion, throughout the sequel we shall consider all limits only for even multipoles l. Under this condition, the asymptotic behaviour of these variances was investigated in [15], where it was shown that, for $q = 3$ and $q \geq 5$

$$\begin{aligned} Var(h_{q;l}) &= (4\pi)^2 q! \int_0^{\pi/2} P_l^q(\cos\vartheta)\sin\vartheta d\vartheta \sim (4\pi)^2 q! \frac{c_q}{l^2} \ , \\ c_q &= \int_0^{\infty} \psi J_0(\psi)^q d\psi \geq 0 \ . \end{aligned}$$

For $q = 2, 4$, the order of magnitude of the corresponding variances is larger (see Lemma 3.2):

$$Var(h_{q;l}) \approx \begin{cases} \frac{1}{l} \ , \text{ for } q = 2 \\ \frac{\log l}{l^2} \ , \text{ for } q = 4 \end{cases} .$$

The constants c_q are immediately seen to be strictly positive for all even values of q. For odd values, we conjecture this to be always the case; a formal proof is left for future research. Therefore, the statement of our main result will entail this condition explicitly:

Theorem 1.1. *For all q such that $c_q > 0$, we have*

$$\frac{h_{2l;q}}{\sqrt{Var(h_{2l;q})}} \to_d \mathcal{N}(0,1), \ \text{as } l \to \infty \ .$$

Theorem 1.1 is a building block for a more general claim: under minimal regularity conditions (i.e. the existence of at least one nonzero coefficient $J_q(G)$ corresponding to a nonzero term $c_q > 0$) we shall have a CLT for square integrable nonlinear functionals of spherical Gaussian eigenfunctions, i.e. for any $\mathcal{G}(f_l) = \int_{\mathbb{S}^2} G(f_l)dx$ such that $\mathbb{E}\mathcal{G}^2(f_l) =: \sigma_G^2(l) < \infty$,

$$\frac{\mathcal{G}(f_{2l}) - \mathbb{E}\left[\mathcal{G}(f_{2l})\right]}{\sigma_G(2l)} \to_d \mathcal{N}(0,1), \text{ as } l \to \infty.$$

In fact we will prove the following stronger result: for $q \neq 4$,

$$d_{TV}\left(\frac{h_{2l;q}}{\sqrt{Var(h_{2l;q})}}, \mathcal{N}(0,1)\right) = O\left(l^{-\delta_q}\right) \tag{2}$$

for some $\delta_q > 0$, where $d_{TV}(.,.)$ denotes the usual total variation distance of random variables

$$d_{TV}(X,Y) = \sup_{A \in \mathcal{B}(\mathbb{R})} \left|\Pr(X \in A) - \Pr(Y \in A)\right|,$$

with the Borel σ-field $\mathcal{B}(\mathbb{R})$. For $q = 4$ the rate of convergence (2) is of slower *logarithmic* order.

1.3. **On the proofs of the main results and outline of the paper.** The ideas behind our main argument can be summarized as follows. Because $f_l(\cdot)$ is a Gaussian field, for any fixed $x \in \mathbb{S}^2$, $H_q(f_l(x))$ belongs to the so-called q-th order Wiener chaos generated by the Gaussian measure governing $f_l(\cdot)$ (see [19]), and so does any linear transform, including $h_{l;q}$. As a consequence of the Nourdin-Peccati Theorem for Stein-Malliavin approximations of Gaussian subordinated random variables (see for instance [18], Theorem 5.26), the following bound holds for each even l, $q \geq 2$:

$$d_{TV}\left(\frac{h_{l;q}}{\sqrt{Var(h_{l;q})}}, \mathcal{N}(0,1)\right) \leq 2\sqrt{\frac{q-1}{3q}\left(\frac{cum_4\left(h_{l;q}\right)}{Var^2(h_{l;q})}\right)}\ ; \tag{3}$$

here, $cum_4\,(Y)$ is the 4th order cumulant of Y, see [18], [19] for more discussion on these points.

The latter bound shows that if we prove that

$$cum_4\left(h_{l;q}\right) = o_{l\to\infty}\left(Var^2(h_{l;q})\right), \tag{4}$$

the Central Limit Theorem for $h_{l;q}$ (where q is fixed and $l \to \infty$) will follow. The bound (4) for $q \geq 5$ is proved along Section 2. Here we first express the 4th order cumulant as an integral over $(\mathcal{S}^2)^4$, using the well-known *Diagram Formula* (see Section 2.1). This will allow us to obtain the desired bound (4) via a tricky multiple application of the Cauchy-Schwartz inequality (Proposition 2.2, whose proof in Section 2.3 takes on most of Section 2); to this end we will divide the domain of integration into the "local" and "global" ones (for the definitions of various ranges see Section 2.3.2). A more detailed explanation of the proof of Proposition 2.2 may be found in Section 2.3.1.

For $q = 3, 4$ proving the bound (4) will require special arguments, presented in Section 3. While the case $q = 3$ was already covered earlier in [12], for $q = 4$ the asymptotic analysis requires the evaluation of so-called Gaunt integrals, connecting moments of Legendre polynomials to Wigner's and Clebsch-Gordan coefficients (see [13], [24]).

Various applications of Theorem 1.1 for the Defect statistics and general polyspectra are discussed in Section 4. The basic idea is to use the Central Limit Theorem for $h_{l;q}$ to establish asymptotic Gaussianity of *finite-order* generic polynomial sequences, and then exploit the expansion (1) of arbitrary functionals (e.g. the Defect) to prove that the distribution of any such functional can be asymptotically approximated by means of a finite-order expansion.

1.3.1. *Some conventions.* In this manuscript, given any two positive sequences a_n, b_n we shall write $a_n \approx b_n$ if there exist two positive constants c_1, c_2 such that $c_1 a_n \leq b_n \leq c_2 a_n$, for all $n = 1, 2, ...$, and $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. Also, we shall write $a_n \ll b_n$ or $a_n = O(b_n)$ when the sequence a_n/b_n is asymptotically bounded; we write $\mu(dx)$ for the usual Lebesgue measure on the unit sphere, so that $\int_{\mathbb{S}^2} \mu(dx) = 4\pi$.

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2. The central limit theorem for $h_{l;q}$, $q \geq 5$

2.1. **Some preliminaries.** We shall now focus on fourth order cumulants for Hermite transforms for arbitrary $q \geq 5$. First we need to recall some well-known background material on the so-called Diagram Formula (see for instance [21], [19], [13] or [17] for recent textbook references).

Fix a set of integers $\alpha_1, ..., \alpha_p$. A diagram is a graph with $(\alpha_1 + ... + \alpha_p)$ vertexes labelled by $1, ..., p$, (α_1 vertexes are labelled by 1, α_2 vertexes are labelled by 2...) such that each vertex has degree 1, i.e. the edges have no common endpoints. We can view the vertexes as belonging to p different rows and the edges may connect only vertexes with different labels, i.e. there are no flat edges on the same row. The set of all such graphs γ is denoted by $\Gamma(\alpha_1, ..., \alpha_p)$; we write $\Gamma_C(\alpha_1, ..., \alpha_p)$ for graphs that are connected, i.e. it is not possible to partition the vertexes into two subsets A and B such that no edge connects a vertex in A with one in B.

Given a diagram γ, let

$$\underline{\eta}(\gamma) = (\eta_{ij}(\gamma)) \in \mathbb{Z}^{\binom{p}{2}}$$

be the vector whose $\binom{p}{2}$ elements $\eta_{ij}(\gamma)$ $(i < j)$ are the number of edges between i and j in the graph γ. The vector $\underline{\eta}$ satisfies

$$\sum_{i,j} \eta_{ij} = 2q, \tag{5}$$

and, moreover, the multiplicities of opposite edges equal, as the following lemma states.

Lemma 2.1. *Let $\gamma \in \Gamma_C(q, q, q, q)$ with arbitrary $q \geq 1$, and $\underline{\eta} = \underline{\eta}(\gamma)$. Let $e = (i, j)$ any edge in γ and $e' = (i', j')$ the unique edge with vertexes disjoint with e, so that $\{i, j, i', j'\} = \{1, 2, 3, 4\}$. Then $\eta_e = \eta_{e'}$.*

Proof. The statement of the lemma follows immediately from the fact that, by reordering the vertexes corresponding to the same label (i.e. in the same row) if necessary, we may assume that all the edges are between vertexes in the same column. □

2.2. **Cumulants of $h_{l;q}$.** With notation as above, the well-known Diagram Formula (see e.g. [19]) provides the following neat expression for the cumulants of our statistics $\{h_{l;q}\}$, namely

$$cum\left\{h_{l;q_1},...,h_{l;q_p}\right\} = cum\left\{\int_{S^2} H_{q_1}(f_l(x_1))dx_1,...,\int_{S^2} H_{q_p}(f_l(x_p))dx_p\right\}$$

$$= \int\limits_{(S^2)^p} cum\left\{H_{q_1}(f_l(x_1)),...,H_{q_p}(f_l(x_p))\right\} dx_1...dx_p$$

$$= \int\limits_{(S^2)^p} \sum_{\gamma\in\Gamma_C(q_1,...,q_p)} \prod_{(i,j)\in\gamma} \mathbb{E}\left\{f_l(x_i)f_l(x_j)\right\} dx_1...dx_p$$

$$= \sum_{\gamma\in\Gamma_C(q_1,...,q_p)} \int\limits_{(S^2)^p} \prod_{(i,j)} \left(\mathbb{E}\left\{f_l(x_i)f_l(x_j)\right\}\right)^{\eta_{ij}(\gamma)} dx_1...dx_p\ ,$$

(recall that $\underline{\eta}(\gamma)$ is the vector of multiplicities of edges in γ as above).

We constraint ourselves to the 4th order cumulants of the form $cum(h_{l;q},h_{l;q},h_{l;q},h_{l;q})$; in this case the expression above simplifies to

$$cum(h_{l;q},h_{l;q},h_{l;q},h_{l;q}) = \sum_{\gamma\in\Gamma_C(q,q,q,q)} M(\underline{\eta}(\gamma)), \tag{6}$$

where for a vector $\eta\in\mathbb{Z}^6_{\geq 0}$ we set

$$M(\underline{\eta}) = \int\limits_{(S^2)^4} \prod_{i<j} P_l(\langle x_i,x_j\rangle)^{\eta_{ij}}dx, \tag{7}$$

with the shortcut $dx = dx_1\cdot\ldots\cdot dx_4$. Since (6) is a finite summation of the $C(\underline{\eta})$ with $\underline{\eta}$ corresponding to a connected diagram $\gamma\in\Gamma_C(q,q,q,q)$ it then remains to bound each of terms as above separately.

We know [14], Lemma 5.2, that for $q\geq 5$,

$$Var(h_{l;q}) \sim \frac{c_q}{l^2}$$

and aim at proving the bound

$$cum(h_{l;q},h_{l;q},h_{l;q},h_{l;q}) = o\left(\frac{1}{l^4}\right),$$

(or stronger); this is sufficient for the central limit theorem by 3. The following proposition implies this estimate bearing in mind (6).

Proposition 2.2. *For $q \geq 5$ and $\underline{\eta} = \underline{\eta}(\gamma)$ with γ a connected diagram, one has*

$$|M(\underline{\eta})| = \begin{cases} O\left(\frac{(\log l)^2}{l^{4\frac{1}{5}}}\right) & q = 5 \\ O\left(\frac{(\log l)^2}{l^{4\frac{2}{7}}}\right) & q = 6 \\ O\left(\frac{(\log l)^{3/2}}{l^{4+\frac{q-6}{2q-3}}}\right) & q \geq 7 \end{cases}.$$

In particular, for every $l \geq 5$,

$$|M(\underline{\eta})| = o\left(\frac{1}{l^4}\right).$$

The methods of the present section also give a useful (i.e. smaller than the square of the variance, sufficient for central limit theorem) upper bound for $M(\underline{\eta})$ with $\underline{\eta}$ corresponding to $q = 4$. It is however weaker than the precise asymptotics of Lemma 3.3, which is the reason for a special treatment we gave to $q = 4$.

Remark 2.3. In fact, it has been recently established by G.Peccati and coauthors that only a subset of terms as above need to be bounded to establish the CLT - those corresponding to circular diagrams (i.e. diagrams, all of whose rows are linked with precisely two other rows, see e.g. [19], Proposition 11.2); the latter were easier to bound by our earlier methods. We will however not use this observation as our present methods can cope with arbitrary terms, though resulting in slightly weaker upper bounds.

2.3. **Proof of Proposition 2.2.**

2.3.1. *On the proof.* One observes that for a "generic" point, each of the 6 terms $P_l(\langle x_i, x_j \rangle)$ in the integrand product in (7) is bounded by $\frac{1}{\sqrt{l}}$ as in (33). Therefore, unless at least one of the 6 angles involved is small, the integrand is of order $O\left(\frac{1}{l^q}\right)$, and the total contribution to the integral in (7) for $q \geq 5$ *should* be of order smaller than $\frac{1}{l^4}$, sufficient for the CLT[1].

To quantify the latter statement, we introduce a small parameter $\varepsilon = \varepsilon(l)$ and separate the domain of integration $(\mathcal{S}^2)^4$ into the set $\mathcal{L}(\varepsilon)$ of points $x \in (\mathcal{S}^2)^4$ all of whose angles $\vartheta(x_i, x_j)$ are greater than ε, and its complement ($\vartheta(x_i, x_j) \leq \varepsilon$ for least one pair of indexes (i, j)), see Section 2.3.2. We call the contribution of the latter subdomain "local" and, analogously, the former's contribution, "global". The global and local contribution are bounded in sections 2.3.3 and 2.3.4, whence it remains to choose the optimal parameter ε the arises from the tradeoff we get from those bounds (Section 2.3.5).

To bound both the global and local case one employs the following observation. It is possible to decrease the number of different angles $\vartheta(x_i, x_j)$ involved in the integral 2.3.2 by applying the Cauchy-Schwartz inequality, multiple times if necessary. The upshot is that in integrals like

$$\int P_l(\cos\theta(\langle x_1, x_2 \rangle)^{s_1} P_l(\cos\theta(\langle x_3, x_4 \rangle)^{s_2} dx$$

[1]This heuristics is not entirely correct, as we believe the correct order of magnitude of the 4th order cumulant to be proportional to $\frac{1}{l^5}$ rather than $\frac{1}{l^q}$. It means that the regime where at least one angle is small does contribute to the integral.

the variables split and we end up evaluating moments of individual Legendre polynomials (that were readily evaluated, see (12) and (11)). Therefore, when we apply the Cauchy-Schwartz inequality to reduce the number of different angles, it is beneficial to pair up angles corresponding to disjoint edges in the diagram. For $q \geq 7$ these kind of observations, combined with uniform estimate (14) valid on $\mathcal{L}(\varepsilon)$ and the small measure of its complement, are sufficient for our purposes. The cases $q = 6, 5$ are a bit more subtle as in these cases we will have to exploit the special structure of vectors $\underline{\eta}$ corresponding to connected diagrams.

2.3.2. *Global and local terms.* Recall that for a $\underline{\eta} \in \mathbb{Z}^6_{\geq 0}$ we defined $C(\underline{\eta})$ as in (7). For a small parameter $\varepsilon = \varepsilon(l) > 0$ we decompose the domain of integration in (7) as following:

$$(\mathbb{S}^2)^4 = \mathcal{L}(\varepsilon) \cup \mathcal{L}(\varepsilon)^c,$$

where

$$\mathcal{L}(\varepsilon) := \left\{ (x_1, \ldots x_4) \in (\mathbb{S}^2)^4 : \, \forall i < j, \, \vartheta(x_i, x_j) > \varepsilon \right\}$$

and $\mathcal{L}(\varepsilon)^c$ is its complement. We then write

$$M(\underline{\eta}) = M_{glob}(\underline{\eta}; \varepsilon) + M_{loc}(\underline{\eta}; \varepsilon),$$

where

$$M_{glob}(\underline{\eta}; \varepsilon) = \int\limits_{\mathcal{L}} \prod_{i<j} P(\langle x_i, x_j \rangle)^{\eta_{ij}} \, dx \tag{8}$$

is the "global contribution" and

$$M_{loc}(\underline{\eta}; \varepsilon) = \int\limits_{\mathcal{L}^c} \prod_{i<j} P(\langle x_i, x_j \rangle)^{\eta_{ij}} \, dx$$

is the "local contribution".

The global and local contributions are bounded in the following couple of lemmas, whose proofs are given in Sections 2.3.3 and 2.3.4 respectively.

Lemma 2.4 ("Global contribution")**.** *As $l \to \infty$, $q \geq 5$ arbitrary, and $\underline{\eta} = \underline{\eta}(\gamma)$ with $\gamma \in \Gamma_C(q, q, q, q)$, we have*

$$|M_{glob}(\underline{\eta}; \varepsilon)| = \begin{cases} O\left(\frac{(\log l)^2}{l^5 \varepsilon} \right) & q = 5 \\ O\left(\frac{(\log l)^2}{l^6 \varepsilon^2} \right) & q = 6 \\ O\left(\frac{\log l}{l^q \varepsilon^{q-3}} \right) & q \geq 7 \end{cases}$$

with constant involved in the "O"-notation depending on q only.

Lemma 2.5 ("Local contribution")**.** *Under the assumptions of Lemma 2.4 and the additional assumption*

$$\varepsilon \gg \frac{1}{l}, \tag{9}$$

one has

$$|M_{loc}(\underline{\eta}; \varepsilon)| = O\left(\frac{(\log l)^{3/2} \varepsilon^{3/2}}{l^3} \right). \tag{10}$$

To prove lemmas 2.4 and 2.5 we will use asymptotics for the 2nd and 4th moments of Legendre polynomials: as $l \to \infty$

$$\int_0^1 P_l(t)^2 dt \sim c_2 \frac{1}{l}. \tag{11}$$

for some $c_2 > 0$ (cf. (31)), and

$$\int_0^1 P_l(t)^4 dt \sim c_4 \frac{\log l}{l^2}. \tag{12}$$

for some $c_4 > 0$ (cf. Lemma 3.2).

2.3.3. *Bounding the global contribution.*

Proof of Lemma 2.4. By the definition of the domain of integration $\mathcal{L}(\varepsilon)$, all the angles involved satisfy $\vartheta(x_i, x_j) > \varepsilon$. Let $\underline{\eta} = \underline{\eta}(\gamma)$ corresponding to a connected diagram $\gamma \in \Gamma_C(q, q, q, q)$.

First we assume that $q \geq 7$. By an easy counting argument, (5), and Lemma 2.1, it may be shown that $\eta_{ij} \geq 2$ for (at least) two disjoint elements of $\underline{\eta}$. With no loss of generality we assume that both $\eta_{12} \geq 2$ and also $\eta_{34} \geq 2$. Since γ is connected, we may also assume $\eta_{13} \geq 1$, $\eta_{24} \geq 1$, again, by the virtue of Lemma 2.1.

Consider the integrand

$$K_{\underline{\eta}}(x_1, \ldots, x_4) = \prod_{i<j} P_l(\langle x_i, x_j \rangle)^{\eta_{ij}} \tag{13}$$

of (8). On $\mathcal{L}(\varepsilon)$ for every $i < j$ we have the uniform upper bound

$$|P_l(\langle x_i, x_j \rangle)| \ll \frac{1}{\sqrt{l\varepsilon}} \tag{14}$$

by (33). Hence

$$|K_{\underline{\eta}}(x_1, \ldots, x_4)| \ll \frac{1}{(l\varepsilon)^{q-3}} P_l(\langle x_1, x_2 \rangle)^2 P_l(\langle x_3, x_4 \rangle)^2 |P_l(\langle x_1, x_3 \rangle) P_l(\langle x_2, x_4 \rangle)|,$$

so that

(15)
$$|M_{glob}(\underline{\eta}; \varepsilon)| \ll \frac{1}{(l\varepsilon)^{q-3}} \int_{(\mathbb{S}^2)^4} P_l(\langle x_1, x_2 \rangle)^2 P_l(\langle x_3, x_4 \rangle)^2 |P_l(\langle x_1, x_3 \rangle) P_l(\langle x_2, x_4 \rangle)| dx$$

(note that at this stage we can afford to increase the domain of integration to the whole of $(\mathbb{S}^2)^4$).

In order to treat the latter integral we apply the Cauchy-Schwartz inequality, dividing the 4 terms into pairs. We team up each edge with its disjoint complement (i.e. the unique edge with no common vertex); it is then possible to split the variables in each of the resulting integrals. This

approach yields:

$$\int\limits_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_2\rangle)^2 P_l(\langle x_3,x_4\rangle)^2 |P_l(\langle x_1,x_3\rangle)P_l(\langle x_2,x_4\rangle)|dx$$

$$\leq \left(\int\limits_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_2\rangle)^4 P_l(\langle x_3,x_4\rangle)^4 dx \right)^{1/2} \cdot \left(\int\limits_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_3\rangle)^2 P_l(\langle x_2,x_4\rangle)^2 dx \right)^{1/2}$$

$$\ll \frac{\log l}{l^3}.$$

by (12) and (11). The statement of the present lemma for $q \geq 6$ then follows upon substituting the latter bound into (15).

For $q = 5, 6$ the same argument remains valid; however the bound it gives is insufficient, and we will need to exploit the special structure of $\underline{\eta}$ in these cases. For $q = 5$, $\underline{\eta}$ has one of the following three shapes (in some order):

$$\underline{\eta} = (4,4,1,1,0,0) \tag{16}$$

or

$$\underline{\eta} = (3,3,2,2,0,0) \tag{17}$$

or

$$\underline{\eta} = (3,3,1,1,1,1), \tag{18}$$

where, by Lemma 2.1, the edges corresponding to $\eta_{ij} = 4, 3, 2$ are disjoint.

For the first case (16), we have (up to reordering $\{x_i\}$)

$$M_{glob}(\underline{\eta};\varepsilon) = \int\limits_{\mathcal{L}(\varepsilon)} P_l(\langle x_1,x_2\rangle)^4 P_l(\langle x_3,x_4\rangle)^4 P_l(\langle x_1,x_3\rangle) P_l(\langle x_2,x_4\rangle) dx,$$

so that the uniform bound (14) yields

$$|M_{glob}(\underline{\eta};\varepsilon)| \ll \frac{1}{l\varepsilon} \int\limits_{\mathcal{L}(\varepsilon)} \int\limits_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_2\rangle)^4 P_l(\langle x_3,x_4\rangle)^4 dx \ll \frac{(\log l)^2}{\varepsilon l^5}$$

by (12). Next, for the second case (17), we bound

$$|M_{glob}(\underline{\eta};\varepsilon)| \ll \frac{1}{\varepsilon l} \int\limits_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_2\rangle)^2 P_l(\langle x_3,x_4\rangle)^2 P_l(\langle x_1,x_3\rangle)^2 P_l(\langle x_2,x_4\rangle)^2 dx,$$

and continue as for $q \geq 7$ by bounding the latter integral using Cauchy-Schwartz, pairing together disjoint edges.

Finally, for $\underline{\eta}$ as in (18), we use (14) to bound

$$|M_{glob}(\underline{\eta};\varepsilon)|$$
$$\ll \frac{1}{l\varepsilon} \int\limits_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_2\rangle)^2 P_l(\langle x_3,x_4\rangle)^2 |P_l(\langle x_1,x_3\rangle)P_l(\langle x_2,x_4\rangle)P_l(\langle x_1,x_4\rangle)P_l(\langle x_2,x_3\rangle)|dx.$$

Pairing up the 4th degree terms in the latter integral together and the rest separately, we apply Cauchy-Schwartz as before, twice for the second term.

This leads to

$$\int_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_2\rangle)^2 P_l(\langle x_3,x_4\rangle)^2 |P_l(\langle x_1,x_3\rangle) P_l(\langle x_2,x_4\rangle) P_l(\langle x_1,x_4\rangle) P_l(\langle x_2,x_3\rangle)| dx$$

$$\leq \left(\int_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_2\rangle)^4 P_l(\langle x_3,x_4\rangle)^4 dx \right)^{1/2} \cdot \left(\int_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_3\rangle)^4 P_l(\langle x_2,x_4\rangle)^4 dx \right)^{1/4} \times$$

$$\times \left(\int_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_4\rangle)^4 P_l(\langle x_2,x_3\rangle)^4 dx \right)^{1/4} ;$$

this implies the statement of the present lemma for $q = 5$ via separation of variables and (12). The proof for $q = 6$ is very similar to the above (but somewhat easier) and thereupon omitted here. In this case η is of one the following 5 forms:

$$\underline{\eta} = (5,5,1,1,0,0), (4,4,2,2,0,0), (4,4,1,1,1,1), (3,3,3,3,0,0), (3,3,2,2,1,1).$$

□

2.3.4. *Bounding the local contribution.*

Proof of Lemma 2.5. We may assume with no loss of generality that

$$\vartheta(x_1,x_2) < \varepsilon$$

in the relevant domain $\mathcal{L}(\varepsilon)^c$. We divide the possibilities for $\underline{\eta}$ into three cases, up to reordering (that $\underline{\eta}$ falls into one of those cases follows from (5) $q \geq 5$, Lemma 2.1 and the the connectedness of γ):

(1) For every $i < j$, $\eta_{ij} > 0$.
(2) We have

$$\eta_{12} = \eta_{34} \geq 4,\ \eta_{13} = \eta_{24} \geq 1,\ \eta_{14} = \eta_{23} = 0.$$

(3) We have

$$\eta_{12} = \eta_{34} \geq 2,\ \eta_{13} = \eta_{24} \geq 2.$$

In case 1 (which gives rise to the dominating term) we may use Cauchy-Schwartz twice to bound

$$|M_{loc}(\underline{\eta};\varepsilon)| \leq \int_{\mathcal{L}^c(\varepsilon)} \prod_{i<j} |P(\langle x_i,x_j\rangle)| dx$$

$$\leq \left(\int_{\mathcal{L}^c(\varepsilon)} P_l(\langle x_1,x_2)^2 P_l(\langle x_3,x_4)^2 P_l(\langle x_1,x_3)^2 dx \right)^{1/2} \times \tag{19}$$

$$\times \left(\int_{\mathcal{L}^c(\varepsilon)} P_l(\langle x_1,x_4)^2 P_l(\langle x_2,x_3)^2 P_l(\langle x_2,x_4)^2 dx \right)^{1/2}$$

and bound each of the two integrals of (19) separately. For the first integral we use Cauchy-Schwartz again:

$$\int_{\mathcal{L}^c(\varepsilon)} P_l(\langle x_1, x_2)^2 P_l(\langle x_3, x_4)^2 P_l(\langle x_1, x_3)^2 dx$$
$$\leq \left(\int_{(\mathbb{S}^2)^4} P_l(\langle x_1, x_2)^4 dx_1 dx_2 P_l(\langle x_3, x_4)^4 dx_3 dx_4 \right)^{1/2} \cdot \left(\int_{\{x_2 : \vartheta(x_1,x_2)<\varepsilon\}} P_l(\langle x_1, x_3 \rangle)^4 dx \right)^{1/2}$$
$$\ll \frac{\log l}{l^2} \cdot \frac{(\log l)^{1/2}}{l} \varepsilon = \frac{(\log l)^{3/2}}{l^3} \varepsilon,$$

by separation of variables and

$$\mu\left(\{x_2 : \vartheta(x_1, x_2) < \varepsilon\}\right) \ll \varepsilon^2.$$

For the other integral of (19), we may use the lack of symmetry w.r.t. variables to improve the bound as follows:

$$\int_{\mathcal{L}^c(\varepsilon)} P_l(\langle x_1, x_4)^2 P_l(\langle x_2, x_3)^2 P_l(\langle x_2, x_4)^2 dx$$
$$\leq \left(\int_{\mathcal{L}^c(\varepsilon)} P_l(\langle x_1, x_4)^4 dx_1 dx_4 \right)^{1/2} \cdot \left(\int_{\{x_2 : \vartheta(x_1,x_2)<\varepsilon\}} P_l(\langle x_2, x_3)^4 dx_2 dx_3 \right)^{1/2} \times$$
$$\times \left(\int_{\{x_2 : \vartheta(x_1,x_2)<\varepsilon\}} P_l(\langle x_2, x_4)^4 dx \right)^{1/2} \ll \frac{(\log l)^{1/2}}{l} \cdot \frac{(\log l)^{1/2}}{l} \varepsilon \cdot \frac{(\log l)^{1/2}}{l} \varepsilon$$
$$= \frac{(\log l)^{3/2}}{l^3} \varepsilon^2.$$

We then obtain the statement of the present lemma in this case upon substituting the last couple of estimates into (19).

In case 2 we use similar ideas (Cauchy-Schwartz twice) to obtain

$$|M_{loc}(\underline{\eta};\varepsilon)| \le \int\limits_{\mathcal{L}^c(\varepsilon)} P_l(\langle x_1,x_2\rangle)^4 P_l(\langle x_3,x_4\rangle)^4 P_l(\langle x_1,x_3\rangle) P_l(\langle x_2,x_4\rangle)dx$$

$$\le \left(\int\limits_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_2\rangle)^4 P_l(\langle x_3,x_4\rangle)^4 P_l(\langle x_1,x_3\rangle) P_l(\langle x_2,x_4\rangle)dx \right)^{3/4} \times$$

$$\times \left(\int\limits_{\mathcal{L}^c(\varepsilon)} P_l(\langle x_1,x_2\rangle)^4 P_l(\langle x_3,x_4\rangle)^4 P_l(\langle x_1,x_3\rangle)^4 P_l(\langle x_2,x_4\rangle)^4 dx \right)^{1/4}$$

$$\ll \frac{(\log l)^{3/2}}{l^3} \cdot \left(\int\limits_{(\mathbb{S}^2)^4} P_l(\langle x_1,x_3\rangle)^4 dx_1 dx_3 \right)^{1/4} \cdot \left(\int\limits_{\{x_2:\vartheta(x_1,x_2)<\varepsilon\}} P_l(\langle x_2,x_4\rangle)^4 dx_2 dx_4 \right)^{1/4}$$

$$\ll \frac{(\log l)^2}{l^4}\sqrt{\varepsilon},$$

which is smaller than the RHS of (10) by (9).

Finally, for case 3 we similarly have

(20)

$$|M_{loc}(\underline{\eta};\varepsilon)| \le \int\limits_{\mathcal{L}^c(\varepsilon)} \left|P_l(\langle x_1,x_2\rangle)^3\right| \left|P_l(\langle x_3,x_4\rangle)^3\right| P_l(\langle x_1,x_3\rangle)^2 P_l(\langle x_2,x_4\rangle)^2 dx$$

$$\le \int\limits_{\mathcal{L}^c(\varepsilon)} P_l(\langle x_1,x_2\rangle)^2 P_l(\langle x_3,x_4\rangle)^2 P_l(\langle x_1,x_3\rangle)^2 P_l(\langle x_2,x_4\rangle)^2 dx$$

$$\le \left(\int\limits_{(\mathbb{S}^2)^2} P_l(\langle x_1,x_2\rangle)^4 dx_1 dx_2 \right)^{1/2} \cdot \left(\int\limits_{(\mathbb{S}^2)^2} P_l(\langle x_3,x_4\rangle)^4 dx_3 dx_4 \right)^{1/2}$$

$$\left(\int\limits_{(\mathbb{S}^2)^2} P_l(\langle x_1,x_3\rangle)^4 dx_1 dx_3 \right)^{1/2} \cdot \left(\int\limits_{\{x_2:\vartheta(x_1,x_2)<\varepsilon\}} P_l(\langle x_2,x_4\rangle)^4 dx_2 dx_4 \right)^{1/2}$$

$$\ll \frac{\log l}{l^2} \cdot \frac{(\log l)^{1/2}}{l} \cdot \frac{(\log l)^{1/2}}{l}\varepsilon = \frac{(\log l)^2}{l^4}\varepsilon,$$

which is less than latter of the expressions on the RHS of (10), again by (9). □

2.3.5. *Concluding the proof of Proposition 2.2.*

Proof. In order to finish the proof of the present proposition it remains to make a suitable choice of the parameter $\varepsilon(l)$ so that both the local and the global contributions as bounded by Lemmas 2.4 and 2.5 will be smaller than the expressions prescribed in Proposition 2.2. The optimal choice for the

arising trade-off is

$$\varepsilon(l) = \begin{cases} \frac{1}{l^{4/5}} & q = 5 \\ \frac{1}{l^{6/7}} & q = 6 \\ \frac{1}{l^{1-\frac{3}{2q-3}}} & q \geq 7 \end{cases},$$

giving the bound in the statement of the present proposition. □

3. The Central Limit Theorem for $h_{l;q}$, $q = 3, 4$.

We shall start from the investigation of total variation bounds for $h_{l;3}$; this result was established in [13], see also [12], [14], but nevertheless we report it here for the sake of completeness. The Lemmas below make some use of so-called Wigner's and Clebsch-Gordan coefficients; see the Appendix for their definition and discussion of some important properties.

Lemma 3.1. *1. The variance of $h_{l;3}$ is given by*

$$\mathbb{E}\left[h_{l;3}^2\right] = 6 \times (4\pi)^2 \begin{pmatrix} l & l & l \\ 0 & 0 & 0 \end{pmatrix}^2 \sim \frac{12}{\pi\sqrt{3}} \frac{(4\pi)^2}{l^2} \ ; \tag{21}$$

2. For the fourth-order cumulant of $h_{l;3}$ we have

$$cum_4(h_{l;3}) \sim \frac{1}{2l+1} \begin{pmatrix} l & l & l \\ 0 & 0 & 0 \end{pmatrix}^4 \ ; \tag{22}$$

3. The following total variation bound holds:

$$d_{TV}\left(\frac{h_{l;3}}{\sqrt{Var(h_{l;3})}}, \mathcal{N}(0,1)\right) = O\left(\frac{1}{\sqrt{l}}\right). \tag{23}$$

As argued in the Introduction, the general strategy for the proofs of our convergence results requires a careful evaluation of the variance and suitable bounds on fourth-order cumulants. For $q = 4$, these computations are provided in the two Lemmas to follow.

Lemma 3.2. *(1) The variance of $h_{l;4}$ is given by*

$$Var\{h_{l;4}\} = 4!(4\pi)^2 \int_0^1 P_l(t)^4 dt$$

(2) As $l \to \infty$ we have

$$\int_0^1 P_l(t)^4 dt \sim \frac{3}{2\pi^2} \frac{\log l}{l^2}.$$

In particular,

$$Var\{h_{l;4}\} \sim 24^2 \frac{\log l}{l^2} \ .$$

Proof. Since

$$\mathbb{E}[h_{l;4}] = 0,$$

the first part of the present lemma follows from

$$Var\{h_{l;4}\} = \mathbb{E}[h_{l;4}^2] = \mathbb{E}\left[\int_{\mathbb{S}^2} H_4(f_l(x))dx \cdot \int_{\mathbb{S}^2} H_4(f_l(y))dy\right]$$

$$= \int_{\mathbb{S}^2\times\mathbb{S}^2} \mathbb{E}\left[H_4(f_l(x))H_4(f_l(y))\right] dxdy = 4! \int_{\mathbb{S}^2\times\mathbb{S}^2} P_l(\langle x,\, y\rangle)^4 dxdy = 4!(4\pi)^2 \int_0^1 P_l(t)^4 dt.$$

To see the second part, we invoke Hilb's asymptotics (32) in the Appendix to write (up to an admissible error, as it is easy to directly check)

$$\int_0^1 P_l(t)^4 dt \sim \frac{1}{l} \int_1^{l+1/2} \sin\left(\frac{\psi}{l+1/2}\right) J_0(\psi)^4 d\psi \sim \frac{1}{l}\int_1^l \frac{\psi}{l} \cdot J_0(\psi)^4 d\psi$$

$$\sim \frac{1}{l^2}\int_1^l \psi \cdot \frac{4}{\pi^2}\frac{\sin(\psi+\pi/4)^4}{\psi^2} d\psi \sim \frac{1}{l^2}\cdot\frac{3}{2\pi^2}\int_1^l \frac{d\psi}{\psi},$$

by the standard asymptotics for the Bessel J_0. □

Lemma 3.3. *As $l \to \infty$, we have*

$$cum_4\{h_{l;4}\} \approx l^{-4}\ .$$

Proof. For our purposes, we need to show that

(24)

$$A_1 := \int_{S^2\times\dots.S^2} P_l(\langle w,z\rangle)P_l^3(\langle w,w'\rangle)P_l(\langle w',z'\rangle)P_l^3(\langle z',z\rangle)dwdzdw'dz' = O\left(\frac{\log^2 l}{l^5}\right)$$

and

(25)

$$A_2 := \int_{S^2\times\dots.S^2} P_l^2(\langle w,z\rangle)P_l^2(\langle w,w'\rangle)P_l^2(\langle w',z'\rangle)P_l^2(\langle z',z\rangle)dwdzdw'dz' \approx \frac{1}{l^4}\ .$$

Concerning the first term, we note that

$$\int_{\mathbb{S}^2} P_l(\langle w,z\rangle)P_l^3(\langle w,w'\rangle)dw$$

$$= \left\{\frac{4\pi}{2l+1}\right\}^4 \int_{\mathbb{S}^2}\left\{\sum_m Y_{lm}(w)\overline{Y}_{lm}(z)\right\}\left\{\sum_{m'} Y_{lm'}(w)\overline{Y}_{lm'}(w')\right\}^3 dw$$

$$= \left\{\frac{4\pi}{2l+1}\right\}^4 \sum_{m_1m_2'm_3'm_4'} \overline{Y}_{lm_1}(z)\overline{Y}_{lm_2'}(w')\overline{Y}_{lm_3'}(w')\overline{Y}_{lm_4'}(w')$$

$$\times \int_{\mathbb{S}^2} Y_{lm_1}(w)Y_{lm_2'}(w)Y_{lm_3'}(w)Y_{lm_4'}(w')dw$$

$$= \frac{(4\pi)^3}{(2l+1)^2}\sum_{m_1m_2'm_3'm_4'} \overline{Y}_{lm_1}(z)\overline{Y}_{lm_2'}(w')\overline{Y}_{lm_3'}(w')\overline{Y}_{lm_4'}(w')$$

$$\times \sum_{LM} (-1)^{L+M} \left\{ C_{l0l0}^{L0} \right\}^2 \frac{C_{lm_1 lm_2'}^{LM} C_{m_3' m_4'}^{L,-M}}{2L+1} .$$

Likewise

$$\int_{\mathbb{S}^2} P_l(\langle w', z' \rangle) P_l^3(\langle z', z \rangle) dz' =$$

$$= \frac{(4\pi)^3}{(2l+1)^2} \sum_{m_1' m_2 m_3 m_4} \overline{Y}_{lm_1'}(w') \overline{Y}_{lm_2}(z') \overline{Y}_{lm_4}(z') \overline{Y}_{lm_4}(z')$$

$$\times \sum_{L'M'} (-1)^{L+M} \left\{ C_{l0l0}^{L'0} \right\}^2 \frac{C_{lm_1' lm_2}^{L'M'} C_{m_3 m_4}^{L',-M'}}{2L'+1} .$$

Note that L is necessarily even here, otherwise the Clebsch-Gordan coefficients $\left\{ C_{l0l0}^{L0} \right\}$ are identically null from (34), whence $(-1)^{L+M} = (-1)^M$. Iterating the same argument twice more we obtain

$$A_1 \approx \sum_{m_1 \ldots m_4''} \sum_{LM} (-1)^M \left\{ C_{l0l0}^{L0} \right\}^2 \frac{C_{lm_1 lm_2'}^{LM} C_{m_3' m_4'}^{L,-M}}{2L+1} \sum_{L'M'} (-1)^{M'} \left\{ C_{l0l0}^{L'0} \right\}^2 \frac{C_{lm_1' lm_2'}^{L'M'} C_{m_3' m_4'}^{L',-M'}}{2L'+1}$$

(26)

$$\times \sum_{L''M''} (-1)^{M''} \left\{ C_{l0l0}^{L''0} \right\}^2 \frac{C_{lm_1' lm_2''}^{L''M''} C_{m_3'' m_4''}^{L'',-M''}}{2L''+1} \sum_{L'''M'''} (-1)^{M'''} \left\{ C_{l0l0}^{L'''0} \right\}^2 \frac{C_{lm_1 lm_2''}^{L'''M''} C_{m_3'' m_4''}^{L''',-M''}}{2L'''+1} .$$

Now, applying iteratively the orthogonality identity (37), we have

$$\sum_{m_2' m_3' m_4'} \sum_{LM} \sum_{L'M'} (-1)^{M+M'} \left\{ C_{l0l0}^{L0} \right\}^2 \frac{C_{lm_1 lm_2'}^{LM} C_{m_3' m_4'}^{L,-M}}{2L+1} \left\{ C_{l0l0}^{L'0} \right\}^2 \frac{C_{lm_1' lm_2'}^{L'M'} C_{m_3' m_4'}^{L',-M'}}{2L'+1}$$

$$= \sum_{m_2'} \sum_{LM} \frac{\left\{ C_{l0l0}^{L0} \right\}^4}{2L+1} \frac{C_{lm_1 lm_2'}^{LM} C_{m_1' m_2'}^{LM}}{2L+1} = \sum_{L} \frac{\left\{ C_{l0l0}^{L0} \right\}^4}{(2l+1)(2L+1)} \delta_{m_1}^{m_1'} .$$

Applying the same argument to the last two terms in (26), we obtain that

$$A_1 \approx \sum_{m_1, m_1', L, L'} \frac{\left\{ C_{l0l0}^{L0} \right\}^4}{2L+1} \frac{\left\{ C_{l0l0}^{L'0} \right\}^4}{2L'+1} \frac{\delta_{m_1}^{m_1'}}{(2l+1)^2}$$

$$= \frac{1}{2l+1} \left\{ \sum_L \frac{\left\{ C_{l0l0}^{L0} \right\}^4}{2L+1} \right\}^2 = O\left(\frac{1}{2l+1} \frac{\log^2 l}{l^4} \right) .$$

Here it is interesting to recall that

$$\sum_L \frac{\left\{ C_{l0l0}^{L0} \right\}^4}{2L+1} = \sum_L (2L+1) \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^4 = \int_0^1 P_l^4(t) dt = \frac{Var\left\{ h_{l;4} \right\}}{4!(4\pi)^2} .$$

see also [14], Lemma A1 and the proof of Lemma 2.3 therein.

Let us now focus on (25). Using again (42) and (41), we have that

$$\int_{\mathbb{S}^2} P_l^2(\langle w, z \rangle) P_l^2(\langle w, w' \rangle) dw$$

$$= \left\{ \frac{4\pi}{2l+1} \right\}^4 \int_{\mathbb{S}^2} \left\{ \sum_m Y_{lm}(w) \overline{Y}_{lm}(z) \right\}^2 \left\{ \sum_{m'} Y_{lm'}(w) \overline{Y}_{lm'}(w') \right\}^2 dw$$

$$= \left\{ \frac{4\pi}{2l+1} \right\}^4 \sum_{m_1 m_2 m_3' m_4'} \overline{Y}_{lm_1}(z) \overline{Y}_{lm_2}(z) \overline{Y}_{lm_3'}(w') \overline{Y}_{lm_4'}(w')$$

$$\times \int_{\mathbb{S}^2} Y_{lm_1}(w) Y_{lm_2}(w) Y_{lm_3'}(w) Y_{lm_4'}(w') dw$$

$$= \frac{(4\pi)^3}{(2l+1)^2} \sum_{m_1 m_2 m_3' m_4'} \overline{Y}_{lm_1}(z) \overline{Y}_{lm_2}(z) \overline{Y}_{lm_3'}(w') \overline{Y}_{lm_4'}(w')$$

$$\times \sum_{LM} \left\{ C_{l0l0}^{L0} \right\}^2 \frac{C_{lm_1 lm_2}^{LM} C_{m_3' m_4'}^{L,-M}}{2L+1} .$$

Iterating the argument, we find that

$$A_2 \approx \sum_{m_1 \ldots m_4''} \sum_{LM} (-1)^M \left\{ C_{l0l0}^{L0} \right\}^2 \frac{C_{lm_1 lm_2}^{LM} C_{m_3' m_4'}^{L,-M}}{2L+1} \sum_{L'M'} (-1)^{M'} \left\{ C_{l0l0}^{L'0} \right\}^2 \frac{C_{lm_1' lm_2'}^{L'M'} C_{m_3' m_4'}^{L',-M'}}{2L+1}$$

(27)

$$\times \sum_{L''M''} (-1)^{M''} \left\{ C_{l0l0}^{L''0} \right\}^2 \frac{C_{lm_1' lm_2'}^{L''M''} C_{m_3'' m_4''}^{L'',-M''}}{2L+1} \sum_{L'''M'''} (-1)^{M'''} \left\{ C_{l0l0}^{L'''0} \right\}^2 \frac{C_{lm_1 lm_2}^{L''M'''} C_{m_3'' m_4''}^{L'',-M'''}}{2L+1} .$$

Again it is sufficient to apply (37) four times to have

$$\sum_{L,L',L'',L'''} \sum_{M,M',M'',M'''} \left\{ C_{l0l0}^{L0} \right\}^2 \left\{ C_{l0l0}^{L'0} \right\}^2$$

$$\times \left\{ C_{l0l0}^{L''0} \right\}^2 \left\{ C_{l0l0}^{L'''0} \right\}^2 \frac{\delta_L^{L'} \delta_{L'}^{L''} \delta_{L''}^{L'''} \delta_{L'''}^{L}}{(2L+1)(2L'+1)} \frac{\delta_M^{M'} \delta_{M'}^{M''} \delta_{M''}^{M'''} \delta_{M'''}^{M}}{(2L''+1)(2L'''+1)}$$

$$= \sum_{LM} \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^8 = \sum_L (2L+1) \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^8 .$$

We shall now need the following result (see again [14], Lemma A.1):

$$\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2 = \gamma_{lL} \times \frac{2}{\pi} \times \frac{1}{L(2l-L)^{1/2}(2l+L)^{1/2}} , \ \frac{1}{2} \leq \gamma_{lL} \leq \frac{8}{5} .$$

Simple manipulations then yield

$$\sum_L (2L+1) \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^8 \leq \sum_{L=0}^{2l-2} \frac{(2L+1)}{L^4(2l-L)^2(2l+L)^2}$$

$$\begin{aligned}
&\leq \sum_{L=0}^{l} \frac{(2L+1)}{L^4(2l-L)^2(2l+L)^2} + \sum_{L=l}^{2l-2} \frac{(2L+1)}{L^4(2l-L)^2(2l+L)^2} \\
&\leq \frac{1}{4l^4} \sum_{L=0}^{l} \frac{(2L+1)}{L^4} + \frac{1}{l^4} \sum_{L=l}^{2l-2} \frac{(2L+1)}{L(2l-L)^2(2l+L)} \\
&= O(l^{-4}) + \frac{1}{l^5} \sum_{L=l}^{2l-2} \frac{1}{(2l-L)^2} = O(l^{-4}) \ ,
\end{aligned}$$

which completes the proof of the upper bound. To prove that this bound is sharp, it suffices to notice that

$$\sum_L (2L+1) \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^8 \geq \begin{pmatrix} l & l & 0 \\ 0 & 0 & 0 \end{pmatrix}^8 = \frac{1}{(2l+1)^4} \ .$$

□

Combining the variance and cumulant results, and exploiting (3), one finally obtains the following result.

Proposition 3.4. *As $l \to \infty$, we have*

$$d_{TV}\left(\frac{h_{4;l}}{\sqrt{Var(h_{4;l})}}, \mathcal{N}(0,1) \right) = O\left(\frac{1}{\log l} \right) \ .$$

4. Applications

4.1. **Polyspectra for spherical random fields.** Let $T(x)$ be a zero-mean Gaussian and isotropic spherical random field, i.e. a measurable application $T : \mathcal{S}^2 \times \Omega \to \mathbb{R}$ such that $T(x) \overset{d}{=} T(gx)$ for all elements of the group of rotations $g \in SO(3)$. It is well-known that the following mean-square representation holds, in the $L^2(dx \times dP)$ sense (see [13], Chapter 5):

$$T(x) = \sum_l T_l(x) \ , \text{ where } \Delta_{\mathbb{S}^2} T_l = -l(l+1) T_l \ .$$

We can hence view the eigenfunctions f_l as the normalized Fourier components of such spherical field, e.g. $f_l(x) := T_l(x)/\sqrt{\mathbb{E}\left[T_l^2(x)\right]}$. In this subsection, we shall consider the central limit theorem for polynomial functionals of the form

$$Z_l = \sum_{q=0}^{Q} b_q \int_{S^2} \{f_l(x)\}^q \, dx \ , \text{ for some } Q \in \mathbb{N} \ , \ b_q \in \mathbb{R} \ .$$

When we view the eigenfunctions f_l as the Fourier components of an isotropic spherical random field, these polynomial statistics cover, for instance, the well-known (moment and cumulant) *polyspectra* of the random field. These are the crucial statistics when searching, for instance, for possible non-Gaussian behaviour in $T(x)$; see for instance [2], [10], and the references therein. Note that there exist deterministic coefficients $\beta_0, ..., \beta_p$ such that

we can write

$$Z_l = \sum_{q=0}^{Q} \beta_q \int_{S^2} H_q(f_{2l}(x))dx = \sum_{q=0}^{Q} \beta_q h_{2l;q} \ .$$

From the results in the previous Section, we have immediately the following

Corollary 4.1. *Assume that $c_q > 0$ for at least one q such that $\beta_q \neq 0$. Then*

$$\frac{Z_l - \mathbb{E}[Z_l]}{\sqrt{Var(Z_l)}} \to_d \mathcal{N}(0,1) \ , \ as\ l \to \infty \ .$$

The proof is immediate in light of Theorem 1.1. Indeed, we are dealing here with a finite linear combination of asymptotically Gaussian random variables, and we recall that for random vectors with components in Wiener chaoses, the multivariate Central Limit Theorem follows from convergence in distribution of the univariate components, see [20]. This result thus extends the Central Limit Theorem provided in [12] for the sequence $\{h_{2l;3}\}$ to polyspectra of arbitrary orders.

It is actually possible to establish stronger results, i.e. to study the rates of convergence in the total variation bound. Rather than focusing on this issue, we move to the Central Limit Theorem for the case of a more general, infinite-order L^2 expansion, as it is the case for the *Defect*.

4.2. **Defect.** In this subsection, we shall focus on one of the most important geometric functionals, namely the Defect. The Defect (or "signed area", see [5]) of a function $\psi : \mathbb{S}^2 \to \mathbb{R}$ is defined as

$$\mathcal{D}(\psi) := \text{meas}\left(\psi^{-1}(0,\infty)\right) - \text{meas}\left(\psi^{-1}(-\infty,0)\right) = \int_{\mathbb{S}^2} \mathcal{H}(\psi(x))dx. \tag{28}$$

Here $\mathcal{H}(t)$ is such that

$$\mathcal{H}(t) = \mathbb{1}_{[0,\infty)}(t) - \mathbb{1}_{(-\infty,0]}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases}, \tag{29}$$

where $\mathbb{1}_A(t)$ is the usual indicator function of the set A, and dx is the Lebesgue measure. In our case, the Defect is the difference between the areas of positive and negative inverse image of f_l, denoted

$$\mathcal{D}_l := \mathcal{D}(f_l).$$

It has been shown by [14] that the following expansion holds, in the $L^2(dP)$ sense

$$\mathcal{D}_l = \sum_{q=1}^{\infty} \frac{J_{2q+1}}{(2q+1)!} h_{l;2q+1} = \sum_{q=1}^{\infty} \frac{(-1)}{\sqrt{2\pi}} \frac{(2q-1)!!}{(2q+1)!} h_{l;2q+1} \ .$$

Trivially $\mathbb{E}[\mathcal{D}_l] = 0$, and from [15] we have that

$$Var(\mathcal{D}_l) = \mathbb{E}\left[\mathcal{D}_l^2\right] \sim \sum_{q=1}^{\infty} a_q \frac{c_{2q+1}}{l^2} + o(l^{-2}) \ , \ a_q = \frac{(2q)!}{4^q (q!)^2 (2q+1)}$$

are the (suitably normalized) Taylor coefficients of arcsin are asymptotic to

$$a_q = \frac{1}{2\sqrt{\pi}q^{3/2}} + o(q^{-3/2}) \text{ , as } q \to \infty$$

by Stirling's formula, and

$$\sum_{q=1}^{\infty} a_q c_{2q+1} > \frac{32}{\sqrt{27}} \ .$$

Note that we know $c_3 > 0$ from (21); any term corresponding to $c_{2q+1} = 0$ can simply be dropped from the expansion, so the rate for this variance is precise. In view of Theorem 1.1, it is then not difficult to prove the following result.

Corollary 4.2. *As $l \to \infty$, we have*

$$\frac{\mathcal{D}_{2l}}{\sqrt{Var(\mathcal{D}_{2l})}} \to_d \mathcal{N}(0,1).$$

Proof. The proof follows a standard argument for nonlinear transforms of Gaussian measures, see for instance [21]. Define

$$\mathcal{D}_{l;m} := \sum_{q=1}^{m} \frac{J_{2q+1}}{(2q+1)!} h_{l;2q+1} \ ;$$

using the trivial inequality $\mathbb{E}\left[(A-B)^2\right] \le 2\mathbb{E}\left[(A-C)^2\right] + 2\mathbb{E}\left[(C-B)^2\right]$, we have that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{\mathcal{D}_l}{\sqrt{Var(\mathcal{D}_l)}} - \frac{\mathcal{D}_{l;m}}{\sqrt{Var(\mathcal{D}_{l;m})}}\right)^2\right] \\
\le\ & 2\mathbb{E}\left[\left(\frac{\mathcal{D}_l}{\sqrt{Var(\mathcal{D}_l)}} - \frac{\mathcal{D}_{l;m}}{\sqrt{Var(\mathcal{D}_l)}}\right)^2\right] + 2\,\mathbb{E}\left[\left(\frac{\mathcal{D}_{l;m}}{\sqrt{Var(\mathcal{D}_l)}} - \frac{\mathcal{D}_{l;m}}{\sqrt{Var(\mathcal{D}_{l;m})}}\right)^2\right] \\
\le\ & \frac{2}{Var(\mathcal{D}_l)}\mathbb{E}\left[(\mathcal{D}_l - \mathcal{D}_{l;m})^2\right] + 2\left(\frac{Var(\mathcal{D}_{l;m})}{Var(\mathcal{D}_l)} + 1 - \frac{2\sqrt{Var(\mathcal{D}_{l;m})}}{\sqrt{Var(\mathcal{D}_l)}}\right) \ .
\end{aligned}$$

Now, using the same argument as in the proof of Proposition 4.2 from [15], pages 9-10, we have that

$$\begin{aligned}
\mathbb{E}\left[(\mathcal{D}_l - \mathcal{D}_{l;m})^2\right] &= \sum_{q=m}^{\infty}\left\{\frac{J_{2q+1}}{(2q+1)!}\right\}^2 \mathbb{E}\left[h_{l;2q+1}^2\right] \\
&= \frac{1}{l^2}\sum_{q=m}^{\infty} a_q c_{2q+1} + o(l^{-2}) \\
&\le \frac{1}{2\sqrt{\pi}}\frac{1}{l^2}\sum_{q=m}^{\infty}\frac{c_5}{q^{3/2}} + o(l^{-2}) = O\left(\frac{1}{l^2\sqrt{m}}\right),
\end{aligned}$$

so that

$$\frac{2}{Var(\mathcal{D}_l)}\mathbb{E}\left[(\mathcal{D}_l - \mathcal{D}_{l;m})^2\right] = O\left(\frac{1}{\sqrt{m}}\right),$$

and

$$\begin{aligned}\frac{Var(\mathcal{D}_{l;m})}{Var(\mathcal{D}_l)}+1-\frac{2\sqrt{Var(\mathcal{D}_{l;m})}}{\sqrt{Var(\mathcal{D}_l)}} &= 2+O(\frac{1}{\sqrt{m}})-2\sqrt{1+O\left(\frac{1}{\sqrt{m}}\right)} \\ &= O\left(\frac{1}{\sqrt{m}}\right).\end{aligned}$$

It follows immediately that

$$\mathbb{E}\left[\left(\frac{\mathcal{D}_l}{\sqrt{Var(\mathcal{D}_l)}}-\frac{\mathcal{D}_{l;m}}{\sqrt{Var(\mathcal{D}_{l;m})}}\right)^2\right]=O\left(\frac{1}{\sqrt{m}}\right). \tag{30}$$

Now, for every fixed m we have

$$\frac{\mathcal{D}_{l;m}}{\sqrt{Var(\mathcal{D}_{l;m})}}\to_d \mathcal{N}(0,1) \text{ as } l\to\infty,$$

and since m can be chosen arbitrarily large, the random variables $\left\{\frac{\mathcal{D}_l}{\sqrt{Var(\mathcal{D}_l)}}\right\}$ must have the same limit, bearing in mind (30) (see e.g. [21]). □

5. Appendix

The Legendre polynomials are defined by Rodrigues' formula

$$P_l(t):=\frac{1}{2^l l!}\frac{d^l}{dt^l}(t^2-1)^l\ .$$

Legendre polynomials are orthogonal with respect to the constant weight $\omega(t)\equiv 1$ on $[-1,1]$, indeed

$$\int_{-1}^{1}P_{l_1}(t)P_{l_2}(t)dt=\frac{2\delta_{l_1}^{l_2}}{2l_1+1}\ ; \tag{31}$$

they also satisfy the well-known Hilb's asymptotics (see e.g. [23], formula (8.21.17) on page 197):

$$P_l(\cos\theta)=\left(\frac{\theta}{\sin\theta}\right)^{1/2}J_0((l+1/2)\theta)+\delta(\theta)\ , \tag{32}$$

where J_0 is the standard Bessel function, and the error term satisfies

$$\delta(\theta)\ll\begin{cases}\theta^{1/2}O(l^{-3/2})\ , & cl^{-1}<\theta<\pi/2\\ \theta^2\ , & 0<\theta<cl^{-1}.\end{cases}$$

In particular, for $\theta\in\left[0,\frac{\pi}{2}\right]$,

$$P_l(\cos\theta)\ll\frac{1}{\sqrt{l\theta}}. \tag{33}$$

Let us now review briefly some notation on Wigner's $3j$ coefficients; see [25], [24] and [4] for a much more detailed discussion, in particular concerning the relationships with the quantum theory of angular momentum and group

representation properties of $SO(3)$. We start from the analytic expression (valid for $m_1+m_2+m_3=0$, see [24], expression (8.2.1.5))

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} := (-1)^{l_1+m_1}\sqrt{2l_3+1}\left[\frac{(l_1+l_2-l_3)!(l_1-l_2+l_3)!(l_1-l_2+l_3)!}{(l_1+l_2+l_3+1)!}\right]^{1/2}$$
$$\times\left[\frac{(l_3+m_3)!(l_3-m_3)!}{(l_1+m_1)!(l_1-m_1)!(l_2+m_2)!(l_2-m_2)!}\right]^{1/2}$$
$$\times\sum_z \frac{(-1)^z(l_2+l_3+m_1-z)!(l_1-m_1+z)!}{z!(l_2+l_3-l_1-z)!(l_3+m_3-z)!(l_1-l_2-m_3+z)!},$$

where the summation runs over all z's such that the factorials are non-negative. This expression becomes much neater for $m_1=m_2=m_3=0$, where we have

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} =$$

(34)
$$\begin{cases} 0 \text{ , for } l_1+l_2+l_3 \text{ odd} \\ (-1)^{\frac{l_1+l_2-l_3}{2}}\frac{[(l_1+l_2+l_3)/2]!}{[(l_1+l_2-l_3)/2]![(l_1-l_2+l_3)/2]![(-l_1+l_2+l_3)/2]!}\left\{\frac{(l_1+l_2-l_3)!(l_1-l_2+l_3)!(-l_1+l_2+l_3)!}{(l_1+l_2+l_3+1)!}\right\}^{1/2} \\ \text{for } l_1+l_2+l_3 \text{ even}\end{cases}.$$

Some of the properties to follow become neater when expressed in terms of the so-called Clebsch-Gordan coefficients, which are defined by the identities (see [24], Chapter 8)

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} = (-1)^{l_3+m_3}\frac{1}{\sqrt{2l_3+1}}C^{l_3m_3}_{l_1-m_1l_2-m_2} \tag{35}$$

$$C^{l_3m_3}_{l_1m_1l_2m_2} = (-1)^{l_1-l_2+m_3}\sqrt{2l_3+1}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}. \tag{36}$$

We have the following orthonormality conditions:

$$\sum_{m_1,m_2} C^{lm}_{l_1m_1l_2m_2}C^{l'm'}_{l_1m_1l_2m_2} = \delta_l^{l'}\delta_m^{m'}, \tag{37}$$

$$\sum_{l,m} C^{lm}_{l_1m_1l_2m_2}C^{lm}_{l_1m_1'l_2m_2'} = \delta_{m_1}^{m_1'}\delta_{m_2}^{m_2'}. \tag{38}$$

Now recall the general formula ([24], eqs. 5.6.2.12-13, or [13], eqs 3.64 and 6.46)

$$\int_{S^2} Y_{l_1m_1}(x)...Y_{l_nm_n}(x)dx$$
$$= \sqrt{\frac{4\pi}{2l_n+1}}\sum_{L_1...L_{n-3}}\sum_{M_1...M_{n-3}}\Bigg[C^{L_1M_1}_{l_1m_1l_2m_2}C^{L_2M_2}_{L_1M_1l_3m_3}...C^{l_n,-m_n}_{L_{n-3}M_{n-3}l_{n-1}m_{n-1}}$$
$$\times\sqrt{\frac{\prod_{i=1}^{n-1}(2l_i+1)}{(4\pi)^{n-1}}}\left\{C^{L_10}_{l_10l_20}C^{L_20}_{L_10l_30}...C^{l_n0}_{L_{n-3}0l_{n-1}0}\right\}\Bigg]. \tag{39}$$

Two important special cases are provided by

$$\int_{\mathbb{S}^2} Y_{lm_1}(x)Y_{lm_2}(x)Y_{lm_3}(x)dx$$

$$= (-1)^{l-m_3}\sqrt{\frac{(2l+1)}{4\pi}}C_{l0l0}^{l0}C_{lm_1lm_2}^{l-m_3}$$

$$= \sqrt{\frac{(2l+1)^3}{4\pi}}\begin{pmatrix} l & l & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l & l \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{40}$$

and

$$\int_{\mathbb{S}^2} Y_{lm_1}(x)Y_{lm_2}(x)Y_{lm_3}(x)Y_{lm_4}(x)dx$$

$$= \frac{(2l+1)}{\sqrt{4\pi}}\sum_L (-1)^{L-M}\left\{C_{l0l0}^{L0}\right\}^2 \frac{C_{lm_1lm_2}^{LM}C_{lm_3lm_4}^{L,-M}}{2L+1}$$

$$= \sqrt{\frac{(2l+1)^4}{4\pi}}\sum_L (2L+1)\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2\begin{pmatrix} l & l & L \\ m_1 & m_2 & M \end{pmatrix}\begin{pmatrix} L & l & l \\ M & m_3 & m_4 \end{pmatrix}. \tag{41}$$

Similarly, the following identities hold:

$$\int_0^1 P_l^3(t)dt = \begin{pmatrix} l & l & l \\ 0 & 0 & 0 \end{pmatrix}^2,$$

$$\int_0^1 P_l^4(t)dt = \sum_{L=0}^{2l}(2L+1)\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2.$$

Finally, we recall the useful identity, valid for all $x_1, x_2 \in \mathbb{S}^2$

$$P_l(\langle x_1, x_2\rangle) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(x_1)\overline{Y}_{lm}(x_2), \tag{42}$$

which allows to express Legendre polynomials in terms of spherical harmonics.

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